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Linear Piecewise Differential Equations with Singular Discontinuity Region

Ricardo Miranda Martins, Samuel Krüger*

Abstract
In this work a discontinuous planar piecewise differential system was considered. In particular, the conditions for the existence of limit cycles in a system with 2 foci and 2 linear fields was studied. We obtain a computational method to construct cycles in such systems and theoretical approaches to verify if the obtained cycles are limit cycles.

Key words:
Dynamical systems, Piecewise differential equations, Limit cycles

Introduction

The theory of Discontinuous Piecewise Differential Systems has been widely studied in the last years, for it is appliccable to many fields of knowledge, as engineering, biology, and mathematics.

In this work we investigate the existence of limit cycles in certain planar piecewise linear systems with 2 foci. Using the *software Mathematica*, cycles were obtained in 2 situations: when the focus is in the origin and when it is in another point of the plane (a virtual focus). We also use Newton-Kantorovich Theorem to verify if given 2-zone linear piecewise system has limit cycles.

Results and Discussion

Consider a planar linear piecewise system where the smooth fields in the i-th quadrant is the solution of the system of ODE's $(x_i',y_i') = (P_i(x_i,y_i),Q_i(x_i,y_i))$, i =1,2,3,4. To simplify the analysis we consider the initial point of each field being upon some real axis.

Define for each $1 \le i \le 4$ the flight time t_i as the smallest positive real value such that $x_i(t_i) = 0$ if i is odd or $y_i(t_i) = 0$ if i is even. We have that even in the case of foci in the origin the calculations of the flight times are very complicated, for they depend on the field considered and the initial point.

Therefore, instead of a generalized approach, we first do a numerical analysis of certain systems,obtaining through computational methods its flight times and its cycles. Afterwards, we studied theoretical methods that can be used to verify if the numerically found cycles are indeed limit cycles. Image 1 portraits the cycles found for (i) the foci in the origin and (ii) for virtual foci.

We remark that if the foci are not in the origin, its field equations are so complicated that its flight times cannot be computed. Therefore we can only find an interval in which the point has to be located and use the Mean Value Theorem.

A problem that arises with this approach is the use of approximations. Indeed, as we compute approximations over approximations, the final cycle is not a continuous curve (for example, the lines that compose the cycles in a linear field are not parallel even though they have the same angular coefficient). Thus we have to use the following theorem to show that the cycles are, indeed, limit cycles:

Theorem (Newton-Kantorovich): Consider a function $f:C\to R^n$ of class C^1 with C a compact of R^n and let C_0 be a convex subset of C. Suppose that there are constants α, β and γ such that (i) $|Df(z0)^{-1} f(z_0)| \le \alpha$ for some $z_0 \in C_0$; (ii) $|Df(z_0)^{-1}| \le \beta$ for some $z_0 \in C_0$; and, (iii) $|Df(z) - Df(z_0)| \le \gamma |z - z_0|$ for every $z, z_0 \in C_0$.
Consider $h = \alpha\beta\gamma$ and the roots $r_{1,2}= (1\pm (1 - 2h)^{1/2})/h\,\alpha$. If $h \le 1/2$ and if $B^*(z_0,r_1) \subset C_0$, then the sequence (z_k), defined as $z_{k+1} = z_k - Df(z_k)^{-1}f(z_k)$, is contained in $B(z_0,r_1)$ and it converges to the only zero of f in $C_0 \cap B(z_0,r_2)$.

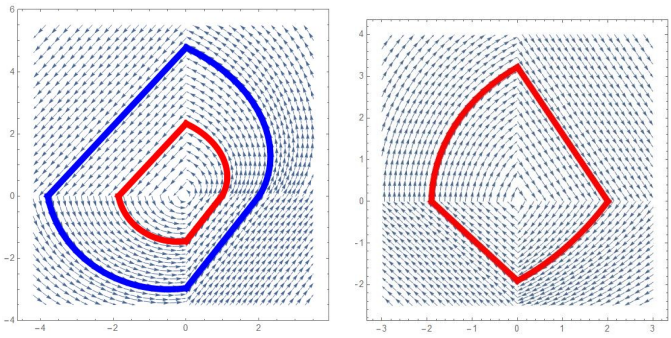

Image 1: Limit cycles obtained with *Mathematica* for the systems (i) $(x_1',y_1')=(0.16x_1-0.28876y_1, 0.28876x_1+0.16y_1)$, $(x_2',y_2')=(-0.79593, -1)$,$(x_3',y_3')=(-0.16x_1-y_1,x_1-0.16y_1)$, $(x_4',y_4')=(1.01478,1.5)$ e (ii) $(x_1',y_1')=(0.621156,-1)$, $(x_2',y_2')=(0.5x_2+2y_2+1,-2x_2+0.5y_2+3)$, $(x_3',y_3')=(-1.5,1.5)$, $(x_4',y_4')=(-0.5x_4+0.45y_4-1, -0.45x_4-0.5y_2-2)$

Conclusions

Using mathematical and computational methods, the existence of limit cycles in planar piecewise systems was studied. In a future project, we will expand the NKT in the 4-zone systems to find its limit cycles.

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[1] Llibre, J., Ponce, E. *Three limit cycles in discontinuous piecewise liner differential systems with two zones.* Dynamical Continuous and Discrete Systems. Ser. B, vol. 19, 325-335, 2012.