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# Limit cycles of piecewise linear differential equations in dimension $n \geq 3$

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## 1 Introduction

The study of piecewise linear differential equations (PWLDE) has been of great interest in the last decades, as they have been proven to be rich in applications. For example, they can predict periodic oscillations in electric circuits [2]. From a purely mathematical point of view, the area has attracted much study as well, for most of the results of the theory of smooth differential equations cannot be applied to the non-smooth case.

However, [4] shows that one can use first integrals (that is, functions that are only constant on the orbits of a differential equation) to describe the orbits in each smooth portion of the plane. If the switching plane is of the crossing kind, then one can concatenate orbits of the different regions of the space. In this work, we use this method to investigate the existence of periodic orbits in three-dimensional PWLDE with one or two switching planes.

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## 2 PWLDE separated by one straight line

**Theorem 1.** *Consider the system*

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}cz\\ -\frac{a}{b}cz\\ -dx\end{pmatrix} \text{ if } x>0; \begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{pmatrix}\gamma z\\ \frac{\alpha}{\beta}\gamma z\\ \delta x\end{pmatrix}, \text{ if } x<0. \tag{1}$$

*If* $b, \beta \neq 0$, $cd > 0$, $\gamma\delta > 0$ *and* $c\gamma > 0$, *then there is a periodi orbit crossing every point* $(0, y_0, z_0)$.

**Corolary 1.** *The system*

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \begin{cases}(z, -\frac{1}{3}z, -x), & \text{if } x>0\\ (4z, -2z, -x), & \text{if } x<0\end{cases} \tag{2}$$

*has an infinite number of periodic orbits.*

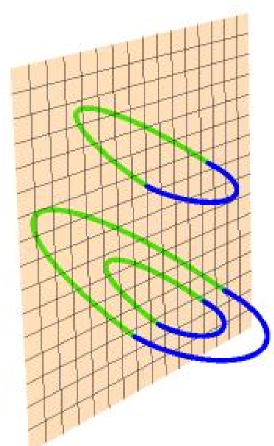

Figure 1: Some periodic orbits of (2). Obtained with *Mathematica*.

## 3 PWLDE separated by two straight lines

In [2], Freire et al studied the symmetric PWLDE with two zones given by

$$\dot{\mathbf{x}} = \begin{cases} A_L\mathbf{x} - \mathbf{b}, & \text{if } x < -1; \\ A_C\mathbf{x} \quad, & \text{if } -1 < x < 1; \\ A_L\mathbf{x} + \mathbf{b}, & \text{if } x > 1. \end{cases} \tag{3}$$

where $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$; $A_L, A_C \in \mathbb{M}_{3\times 3}(\mathbb{R})$; and $\mathbf{b} \in \mathbb{R}^3$.

It was shown that this kind of system has a focus-center bifurcation:

**Theorem 2.** *[2] Let $t, d, m$ (resp. $T, D, M$) denote respectively the trace, determinant and sum of principal minors of order two of $A_L$ (resp. $A_C$) and assume that $M > 0$. Define $T_0 = M/D$ and $\gamma = DM - Dm + dM - tM^2$. Then:*

*a) If $T = T_0$, (3) has a center on the central region;*

*b) (3) has a symmetric limit cycle of three regions if $(T - T_0)\gamma > 0$, for $T - T_0$ sufficiently small.*

Our aim was to investigate these systems with their first integrals. First, we give a result when $A_C = A_L$:

**Theorem 3.** *Consider system (3) with*

$$A_C = A_L = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ -2 & 2 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} q - r \\ r - p \\ p - q \end{bmatrix}, \text{ and } \quad p, q, r \in \mathbb{R}$$

*and let $\boldsymbol{p} = (1, y_0, z_0)$ with $y_0 \geq 0$ be a point on the switching plane.*

*a) If $y_0 = 0$ and $q \geq r$, then the orbit through $\boldsymbol{p}$ is periodic and contained in the central region;*

*b) If $y_0 = 0$ and $q < r$, then none orbit through $\boldsymbol{p}$ is periodic;*

*c) If $y_0 > 0$ and $q = r$, then the orbit through $\boldsymbol{p}$ is periodic and of three regions;*

*d) If $y_0 > 0$ and $q \neq r$, then none orbit through $\boldsymbol{p}$ is periodic.*

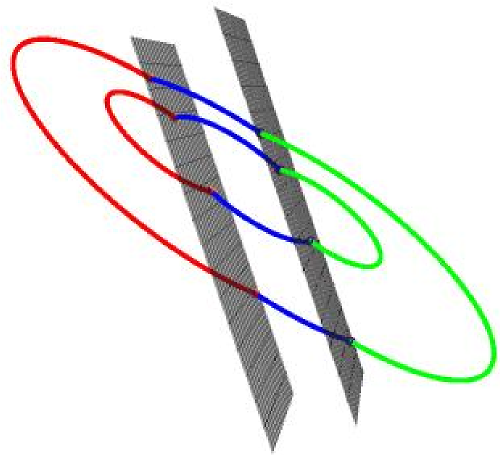

Figure 2: Some periodic orbits of Theorem 3 with parameters $p = -1, q = 2, r = 2$. Obtained with *Mathematica*.

Due to the complexity of expressions of the first integrals, we were not able to produce an example of periodic orbit when $A_C$ and $A_L$ are not identical matrices. However, we were able to produce a criterion for existence of limit cycles from the first integrals of the central system:

**Theorem 4.** *If*

$$\dot{\boldsymbol{x}} = A_C \boldsymbol{x}$$

*has two functionally independent quadratic polynomials as first integrals, then* (3) *admits no limit cycle of three regions.*

## References


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