INTRODUCTION TO MALLIAVIN CALCULUS AND APPLICATIONS

Jamil G. de Abreu Jr (Bolsista – Pibic/CNPq)   Pedro J. Catuogno (Orientador)

IMECC (Instituto de Matemática, Estatística e Computação Científica) – UNICAMP

Keywords: Probability, Stochastic Calculus, Financial Market

Introduction

Since the celebrated papers by Black and Scholes [Bla-Sch] and Merton [Mer] the idea of using stochastic calculus for modeling prices of risky assets (share prices of stock, stock indices such as the Dow Jones, Nikkei or DAX, foreign exchange rates, interest rates, etc.) has been generally accepted. This led to a new branch of applied probability theory, the field of mathematical finance. It is a symbiosis of stochastic modelling, economic reasoning and practical financial engineering.

In what follows, we give some definitions and then we collect some basic facts needed for defining stochastic integrals. A stochastic process is a natural model for describing the evolution of real-life processes, objects and systems in time and space. One particular stochastic process, namely Brownian Motion, has motivated most of the development of stochastic calculus.

Brownian Motion

1.1 Definition. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $\mathcal{B}$ denote the borel subsets of $\mathbb{R}^n$. A *random variable* is a measurable function $X : \Omega \to \mathbb{R}^n$, i.e, a function such that $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}$.

1.2 Definition. A *stochastic process* X is a collection of random variables $\{X_t : t \in T\}$ defined on some probability space Ω. For each fixed $\omega \in \Omega$, the map $\omega \mapsto X_t(\omega)$ is a *trajectory* or a *sample path* of the process.

1.3 Definition. A stochastic process $B = \{B_t : t \geqslant 0\}$ is called a (*standard*) *Brownian motion* or a *Wiener process* if the following conditions are satisfied:
(i) $B_0 = 0$, i.e., it starts at zero;
(ii) $t \mapsto B_t(\omega)$ is continuous almost surely;
(iii) for all times $0 < t_1 < t_2 < \cdots < t_n$, the increments $B_{t_1}, B_{t_2} - B_{t_1}, B_{t_n} - B_{t_{n-1}}$ are independent;
(iv) $B_t - B_t$ is $N(0, t-s)$ for all $t \geqslant s \geqslant 0$.

1.4 Definition. Let X be a real-valued continuous function on $[0, T]$. If the limit

$$\langle X \rangle_t = \lim_{n\to\infty} \sum_{\pi_n \ni t_i \leqslant t} (X_{t_i} - X_{t_{i-1}})^2$$

exists, then $t \mapsto \langle X \rangle_t$ is called the *quadratic variation* of X.

1.5 Proposition. $\langle B \rangle_t = t$ almost surely.
From this, we can prove the following result.

1.6 Theorem. Let $B = \{B_t : t \geqslant 0\}$ as in definition 1.3. Then, on any finite interval

$$P\left\{\sup_{\pi} \sum_{t_i \in \pi} |B_{t_i} - B_{t_{i-1}}| < \infty\right\} = 0,$$

i.e, the sample paths of Brownian motion has unbounded variation almost surely.

1.7 Definition. A function $X : [0, T] \to \mathbb{R}$ is *Holder continuous with exponent* γ *at the point* t_0 if there exists a constant K such that

$$|X_t - X_{t_0}| \leqslant K|t - t_0|^\gamma \text{ for all } t \in [0, T].$$

It follows from a very well know theorem due to Kolmogorov that *for almost all* ω *and any* $T > 0$, *the sample path* $t \mapsto B_t(\omega)$ *is uniformly Holder continuous on* $[0, T]$ *for each exponent* $0 < \gamma < \frac{1}{2}$.
On the other hand, the following holds.

1.8 Theorem. For each $\frac{1}{2} < \gamma \leqslant 1$ and almost every ω, $t \mapsto B_t(\omega)$ is nowhere Holder continuous with exponent γ. In particular, *for almost every* ω, *the sample path* $t \mapsto B_t(\omega)$ *is nowhere differentiable*.

Stochastic Integral

We recall some basic facts from classical analysis in order to understand the main difficulties in defining an integral such as $\int_0^1 B_t dB_t$.

2.1 Definition. We say that $X : [0, T] \to \mathbb{R}$ has *bounded variation* if

$$\sup_{\pi} \sum_{t_i \in \pi} |X_{t_i} - X_{t_{i-1}}| < \infty$$

where the supremum is taken over all partitions π of $[0, T]$.

2.2 Definition For $X, Y : [0, T] \to \mathbb{R}$, we define the *Riemann-Stieltjes integral of* X *with respect to* Y *on* $[0, T]$ as

$$\int_0^T X_t dY_t = \lim_{n\to\infty} \sum_{t_i \in \pi_n} X_{t_i^*}(Y_{t_i} - Y_{t_{i-1}})$$

when such limit exists as $||\tau_n|| \to 0$, and it is independent of the choice of the sequence (τ_n) and their intermediate points t_i^*.

One sufficient condition for the existence of such integral is that X be continuous and Y has bounded variation. Indeed, there exists weaker conditions, not very well known, under which we could define an integral such as $\int_0^1 f(t) dB_t(\omega)$ for a deterministic function f satisfying some nice conditions. However, if we insist in trying to define $\int_a^b B_t dB_t$ by taking limits of Riemann sums, we get the following result.

2.3 Proposition Let (π_n) be a sequence of partitions of $[0, T]$ such that $||\tau_n|| \to 0$ and let $0 \leqslant \lambda \leqslant 1$ be fixed. Then

$$\lim_{n\to\infty} \sum_{t_i \in \pi_n} B_{t_i^*}(B_{t_i} - B_{t_{i-1}}) \overset{L^2(\Omega)}{=} \frac{B_T^2}{2} + \left(\lambda - \frac{1}{2}\right) T.$$

It turns out that Ito's definition of $\int_0^T B_t dB_t$ corresponds to the choice $\lambda = 0$, that is

$$\int_0^T B_t dB_t = \frac{B_T^2}{2} - \frac{T}{2}.$$

An alternative definition, due to Stratonovich, takes $\lambda = \frac{1}{2}$, so that

$$\int_0^T B_t \circ dB_t = \frac{B_T^2}{2}.$$

Ito's Formula

Recall Definition 1.4. Since $t \mapsto \langle X \rangle_t$ is a positive and monotone function, the integral $\int_0^T f(t) d\langle X \rangle_t$ is well-defined in the Riemann-Stieltjes sense. We now establish the Ito's formula, one of the most important tools for computing stochastic integrals.

3.1 Theorem (Ito's formula). Let $X : [0, T] \to \mathbb{R}$ be continuous with continuous quadratic variation $\langle X \rangle_t$, and $f \in C^2(\mathbb{R})$ a twice continuously differentiable real function. Then for each t

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2}\int_0^t f''(X_s) d\langle X \rangle_s$$

where

$$\int_0^t f'(X_s) dX_s := \lim_{n\to\infty} \sum_{\pi_n \ni t_i \leqslant t} f'(X_{t_i})(X_{t_{i+1}} - X_{t_i})$$

is, by definition, the *Ito integral of* $f'(X_t)$ *with respect to* X_t. In short notation

$$df(X) = f'(X)dX + \tfrac{1}{2} f''(X) d\langle X \rangle.$$

3.2 Example. Take $f(x) = x^2$ and $X = B$ to be Brownian motion. Then Ito's formula implies

$$B_t^2 = \underbrace{B_0^2}_{0} + 2\int_0^t B_s dB_s + \underbrace{\int_0^t d\langle B \rangle_s}_{t}$$

so

$$\int_0^t B_s dB_s = \frac{B_t^2}{2} - \frac{t}{2}.$$

3.3 Proposition. For $f \in C^1$ the quadratic variation of $f(X_t)$ is

$$\langle f(X) \rangle_t = \int_0^t f'(X_s)^2 d\langle X \rangle_s.$$

3.4 Definition. Let X, Y be real-valued continuous functions on $[0, T]$ with continuous quadratic variations $\langle X \rangle$ and $\langle Y \rangle$. If the limit

$$\langle X, Y \rangle_t = \lim_{n\to\infty} \sum_{\pi_n \ni t_i \leqslant t} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}})$$

exists, then $t \mapsto \langle X, Y \rangle_t$ is called the *covariation* of X and Y.

Let now $X = (X^1, \ldots, X^d) : [0, T] \to \mathbb{R}^d$ be continuous with continuous covariation

$$\langle X^k, X^l \rangle_t = \begin{cases} \langle X^k \rangle_t & , k = l \\ \frac{1}{2}[\langle X^k + X^l \rangle_t - \langle X^k \rangle_t - \langle X^l \rangle_t] & , k \neq l \end{cases}.$$

3.5 Theorem (*d*-dimensional Ito's formula). For $f \in C^2(\mathbb{R}^d)$ one has

$$f(X_t) = f(X_0) + \overbrace{\int_0^t \nabla f(X_s) dX_s}^{\text{Ito integral}} + \frac{1}{2}\sum_{k,l=1}^{d} \int_0^t f_{x_k, x_l}(X_s) d\langle X^k, X^l \rangle_s.$$

or, in differential form

$$df(X_t) = \underbrace{\langle \nabla f(X_t), dX_t \rangle}_{\text{scalar product}} + \frac{1}{2}\sum_{k,l} \frac{\partial^2 f}{\partial x_k \partial x_l}(X_t) d\langle X^k, X^l \rangle_t$$

where

$$\int_0^t \nabla f(X_s) dX_s := \lim_{n\to\infty} \sum_{\pi_n \ni t_i \leqslant t} \underbrace{(\nabla f(X_{t_i})(X_{t_i} - X_{t_{i-1}}))}_{\text{scalar product}}.$$

Application to Financial Markets

We consider a financial market with only one security without interest and divided payments. This market is modelled as follows:

- $(\Omega, (\mathcal{F}_t)_{t\geqslant 0}, P)$ is a filtered probability space, i.e., $(\mathcal{F}_t)_{t\geqslant 0}$ is a family of σ-algebras with $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leqslant t$, representing the information available at time t.
- $X_t = X_t(\omega)$ is the *price process of the security adapted to the filtration* $(\mathcal{F}_t)$, i.e., X_t is $\mathcal{F}_t$-measurable for all $t \geqslant 0$.
- $\phi_t = \phi_t(\omega)$ is another stochastic process adapted to $(\mathcal{F}_t)$, called a *portfolio strategy*. It denotes the number of shares of the security held by an investor at time t in state ω. Adaptation to $(\mathcal{F}_t)$ means that the investment decision can only be based on the information available at time t.

Given the portfolio strategy ϕ_t, the value of the portfolio at time t is of the form

$$V_t = \phi_t X_t + \eta_t = V(X_t, t)$$

where η_t is money account, yielding no interest.

A portfolio strategy (p.s.) is called *self-financing* if, after an initial investment $V_0 = \eta_0$, all changes in the value of the portfolio V_t are only due to the accumulated gains (or losses) resulting from price changes of X_t. Formally this means

4.1 Definition. The p.s. ϕ_t is *self-financing* if

$$V_t = V_0 + \int_0^t \phi_s dX_s,$$

or, in short notation, $dV = \phi dX$.

Applying Ito's formula to the value process V yields

$$\begin{aligned} dV &= V_x dX + \dot{V} dt + \tfrac{1}{2} V_{xx} d\langle X \rangle \\ &= \phi dX + \dot{V} dt + \tfrac{1}{2} V_{xx} d\langle X \rangle. \end{aligned}$$

Hence ϕ_t is self-financing if, and only if, V satisfies the differential equation

$$\dot{V} dt + \tfrac{1}{2} V_{xx} d\langle X \rangle = 0$$

for all $t > 0$, where $\dot{V} = \frac{\partial}{\partial t} V(x, t)$.

Referências

[Bla-Sch] Black, F. and Scholes, M., The Pricing of Options and Corporate Liabilities, *J. Political Economy* **81**, 635-654, 1973.

[Mer] Merton, R.C., Theory of Option Pricing. *Bell. J. Econ. Manag. Sci.* **4**, 141-183, 1973.

[Evans] Evans, L. C., An Introduction to Stochastic Differential Equations - Version 1.2., http://math.berkeley.edu/ evans/SDE.course.pdf

[Sond] Sondermann, D., Introduction to Stochastic Calculus for Finance, Springer, 2006.